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Conformal symmetry on the instanton moduli space

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Abstract

The conformal symmetry on the instanton moduli space is discussed using the ADHM construction, where a viewpoint of 'homogeneous coordinates' for both the spacetime and the moduli space turns out to be useful. It is shown that the conformal algebra closes only up to global gauge transformations, which generalizes the earlier discussion by Jackiw *et al*. An interesting five-dimensional interpretation of the $SU(2)$ single-instanton is also mentioned.

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We begin our discussion with a brief review of some relevant aspects. For investigating conformal properties on the four-dimensional spacetime, as well as for the study of instanton solutions in gauge field theory and the expression of ADHM construction, it is convenient to introduce a quaternionic notation for the (Euclidean) spacetime coordinates,

$$x \equiv x^n \sigma_n, \qquad \bar{x} \equiv x^n \bar{\sigma}_n = x^\dagger, \tag{1}$$

where $\sigma_n \equiv (\mathrm{i}\vec{\tau}, 1_2)$ and τ^i, $i = 1, 2, 3$ are the three Pauli matrices, and the conjugate matrices $\bar{\sigma}_n \equiv (-\mathrm{i}\vec{\tau}, 1_2) = \sigma_n^\dagger$. The action of the whole conformal group on the quaternionic coordinates can be written elegantly as

$$x \to \tilde{x} = (Ax + B)(Cx + D)^{-1}, \qquad \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1, \tag{2}$$

where A, B, C and D are quaternions viewed here as 2×2 matrices. This fractional linear form of conformal transformation enables us to introduce a kind of 'homogeneous quaternionic coordinates' whose relation to the ordinary quaternionic coordinates is

$$X \equiv \begin{pmatrix} u \\ v \end{pmatrix}, \qquad x = uv^{-1}. \tag{3}$$

Obviously, now the quaternionic matrix

$$\mathcal{C} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4}$$

acts linearly on X, which forms $SL(2,\mathbb{H})$, the Euclidean version of twistor representation of the conformal group.

As is well known, the ADHM construction [1–3] (for the self-dual instantons, without loss of generality) is described by the basic object Δ, an $(N+2k)\times 2k$ matrix linear in the spacetime coordinates,

$$\Delta = a + bx \equiv a + b(x\otimes 1_k), \tag{5}$$

with the complex-valued constant matrices (in the canonical form)

$$a = \begin{pmatrix} K \\ \underline{a} \end{pmatrix}, \qquad b = \begin{pmatrix} 0_{N\times 2k} \\ 1_{2k} \end{pmatrix}. \tag{6}$$

The degrees of freedom in a constitute an overcomplete set of collective coordinates on the instanton moduli space. Roughly speaking, $\underline{a}$ contains the information about positions of the instantons, while K contains that about sizes and gauge orientations of them. Transformation (2) gives [4]

$$\Delta(\tilde{x}; a, b) = \Delta(x; aD + bB, aC + bA)(Cx + D)^{-1}. \tag{7}$$

It can be seen that the $(Cx+D)^{-1}$ factor is not essential, and if $\Delta(x; a, b)$ satisfies the ADHM constraints, so does $\Delta(x; aD+bB, aC+bA)$. Thus, we have the following conformal transformation of the ADHM matrices:

$$a \to a' = aD + bB, \qquad b \to b' = aC + bA. \tag{8}$$

In fact, this transformation can be seen more directly from the following form of Δ,

$$\Delta = \mathcal{A}X, \qquad \mathcal{A} \equiv (b \quad a), \tag{9}$$

where X takes the standard form

$$X = \begin{pmatrix} x \\ 1_2 \end{pmatrix} \equiv \begin{pmatrix} x \\ 1_2 \end{pmatrix} \otimes 1_k, \tag{10}$$

and the action (4) on X as the anti-action on $\mathcal{A}$, which can be regarded as the 'homogeneous collective coordinates' on the instanton moduli space.

To achieve the canonical form (6) from equation (8), one has to make a symmetry transformation of the ADHM construction, which takes b' back to the canonical form,

$$\Lambda b' G = b, \tag{11}$$

where $\Lambda \in \mathrm{U}(N+2k)$ and $G \in GL(k,\mathbb{C})$, while a' becomes

$$\tilde{a} = \Lambda a' G. \tag{12}$$

Thus, the next step is to solve Λ and G. First we have

$$(\Lambda b' G)^\dagger (\Lambda b' G) = G^\dagger b'^\dagger b' G = b^\dagger b = 1_{2k}. \tag{13}$$

The solution of this equation is obviously not unique, but one can naturally take a special one[1]:

$$G = G^\dagger = (b'^\dagger b')^{-1/2}. \tag{14}$$

It can be seen that this G is in $GL(k,\mathbb{C})$, since

$$b'^\dagger b' = C^\dagger a^\dagger a C + A^\dagger \underline{a} C + (A^\dagger \underline{a} C)^\dagger + A^\dagger A \tag{15}$$

[1] The square-root operation in equation (14) is well defined, guaranteed by that $b'^\dagger b'$ is a positive-semidefinite Hermitian matrix. However, the validity of the inversion operation here relies on the positive definiteness of $b'^\dagger b'$. It can be seen that this is the case in general.

and the matrix a satisfies the ADHM constraints. Furthermore, we have

$$(\Lambda b' G)(\Lambda b' G)^\dagger = \Lambda b'(b'^\dagger b')^{-1} b'^\dagger \Lambda^\dagger = bb^\dagger = \begin{pmatrix} 0_N & 0 \\ 0 & 1_{2k} \end{pmatrix}. \tag{16}$$

To deal with matrix equations, especially for non-square matrices, matrix partition is a rather general tool. In fact, the above equation can be solved by the following decomposition,

$$\Lambda^\dagger = (\Lambda_{0\,(N+2k)\times N} \quad \Lambda_{1\,(N+2k)\times 2k}), \tag{17}$$

where Λ_0 and Λ_1 are the orthonormal zero-mode and nonzero-mode matrices of $b'^\dagger$, respectively.

Suppose that P is the projection operator onto the vector space of nonzero modes of $b'^\dagger$. It is easy to see that

$$\Lambda_1 \Lambda_1^\dagger = P = b'(b'^\dagger b')^{-1} b'^\dagger. \tag{18}$$

So we can take

$$\Lambda_1 = b'(b'^\dagger b')^{-1/2}. \tag{19}$$

There exists the completeness relation

$$\Lambda_0 \Lambda_0^\dagger + P = 1_{N+2k}. \tag{20}$$

Again introduce the matrix partition

$$\Lambda_0 = \begin{pmatrix} \lambda_{0\,N\times N} \\ \underline{\Lambda}_{0\,2k\times N} \end{pmatrix} \tag{21}$$

and recall from equation (6) the partition

$$b' = \begin{pmatrix} KC \\ \underline{a}C + A \end{pmatrix} \equiv \begin{pmatrix} L \\ \underline{b}' \end{pmatrix}. \tag{22}$$

Now the completeness relation (20) becomes

$$\begin{pmatrix} \lambda_0 \lambda_0^\dagger + L(b'^\dagger b')^{-1} L^\dagger & \text{h.c.} \\ \underline{\Lambda}_0 \lambda_0^\dagger + \underline{b}'(b'^\dagger b')^{-1} L^\dagger & \underline{\Lambda}_0 \underline{\Lambda}_0^\dagger + \underline{b}'(b'^\dagger b')^{-1} \underline{b}'^\dagger \end{pmatrix} = 1_{N+2k}. \tag{23}$$

This matrix equation is easily solved by (a special solution)

$$\lambda_0 = \lambda_0^\dagger = [1 - L(b'^\dagger b')^{-1} L^\dagger]^{1/2}, \tag{24}$$

$$\underline{\Lambda}_0 = -\underline{b}'(b'^\dagger b')^{-1} L^\dagger {\lambda_0}^{-1}. \tag{25}$$

Here equation (24) is actually the 'singular gauge' when solving the matrix equation for the zero-mode matrix of Δ [3], while all other solutions of equation (20) are related to equation (24) by $U(N)$ gauge transformations.

From the above discussion we obtain

$$\Lambda = \begin{pmatrix} \lambda_0 & -{\lambda_0}^{-1} L(b'^\dagger b')^{-1} \underline{b}'^\dagger \\ (b'^\dagger b')^{-1/2} L^\dagger & (b'^\dagger b')^{-1/2} \underline{b}'^\dagger \end{pmatrix}. \tag{26}$$

It is straightforward to check that equation (11) is satisfied for this Λ and G in equation (14). Recall from equation (6) the matrix partition

$$a' = \begin{pmatrix} KD \\ \underline{a}D + B \end{pmatrix} \equiv \begin{pmatrix} M \\ \underline{a}' \end{pmatrix}. \tag{27}$$

So from equation (12) we have

$$\tilde{a} = \begin{pmatrix} \lambda_0 M - {\lambda_0}^{-1} L(b'^\dagger b')^{-1} \underline{b}'^\dagger \underline{a}' \\ (b'^\dagger b')^{-1/2} (L^\dagger M + \underline{b}'^\dagger \underline{a}') \end{pmatrix} (b'^\dagger b')^{-1/2}. \tag{28}$$

For the identity transformation, i.e., $A = D = 1_2$ and $B = C = 0$, it is easy to see that

$$\lambda_0 = 1, \qquad \tilde{a} = \begin{pmatrix} K \\ \underline{a} \end{pmatrix} = a, \tag{29}$$

as expected. As the simplest nontrivial example, let us consider the inversion transformation $\tilde{x} = x^{-1}$, i.e., $A = D = 0$ and $B = C = 1_2$. Now we have $a' = b$ and $b' = a$, so the transformation (28) can be simplified to

$$\tilde{a} = \begin{pmatrix} -(1 - WW^\dagger)^{-1/2} W \\ 1_{2k} \end{pmatrix} G\underline{a}^\dagger G, \tag{30}$$

where $G = (a^\dagger a)^{-1/2}$ and $W = KG$. The composition of two successive inversions (30) is an identity transformation on a (for nonsingular points of the moduli space), but it needs a little effort to verify this fact, where two identities are useful: one is[2]

$$\underline{a}(a^\dagger a)^{-1} K^\dagger [1 - K(a^\dagger a)^{-1} K^\dagger]^{-1} K(a^\dagger a)^{-1} \underline{a}^\dagger + \underline{a}(a^\dagger a)^{-1} \underline{a}^\dagger = 1, \tag{31}$$

the other is

$$1 - \tilde{K}(\tilde{a}^\dagger \tilde{a})^{-1} \tilde{K}^\dagger = 1 - K(a^\dagger a)^{-1} K^\dagger \tag{32}$$

with $\tilde{K}$ defined by partition (6) of equation (31).

Generically, it can be shown that the conformal transformations (28) of a cannot form an exact realization of the conformal group. However, they do form a 'projective' realization. Suppose that the transformation matrices (Λ, G) and (Λ', G') correspond to two successive conformal transformations $\mathcal{C}$ and $\mathcal{C}'$ on X, respectively. We have for the successive transformations on the moduli space

$$(b \quad \tilde{\tilde{a}}) = \Lambda (b \quad \tilde{a}) \mathcal{C} G = \Lambda \Lambda' (b \quad a) \mathcal{C}' \mathcal{C} G' G, \tag{33}$$

where the commutativity of G' and $\mathcal{C}$ has been used. At the same time, we have for the conformal transformation $\mathcal{C}'' = \mathcal{C}'\mathcal{C}$

$$(b \quad \hat{a}) = \Lambda'' (b \quad a) \mathcal{C}'' G''. \tag{34}$$

It can be seen that equation (11) fixes Λ and G up to the following transformation

$$\Lambda \to \begin{pmatrix} \mathcal{U} & 0 \\ 0 & 1_2 \otimes u \end{pmatrix} \Lambda, \qquad G \to G u^\dagger, \tag{35}$$

where $\mathcal{U} \in U(N)$ and $u \in \mathrm{U}(k)$. Thus, we conclude from equations (33), (34) that $(\Lambda\Lambda', G'G)$ and (Λ'', G'') must be related by transformation (35). Considering the action (12) of Λ and G on a, u is the so-called auxiliary transformation and actually has no effect on the instanton moduli space; but the $SU(N)$ part of $\mathcal{U}$ acts nontrivially on the instanton moduli space as a global gauge transformation. The latter fact is in accord with the early work of Jackiw and Rebbi [6], who find that conformal transformations of the single-instanton configuration are in some sense generically accompanied by gauge transformations. If we disregard the global gauge orientation of the instanton configuration, transformations of the instanton moduli space induced by equation (28) will form a realization of the conformal group.

In some simple cases, one can easily go beyond the singular gauge. Here we use the inversion of $SU(2)$ single-instanton as an example. For an $SU(2)$ single-instanton with size ρ and position c, we have the ADHM matrix

$$a = \begin{pmatrix} 1_2 \otimes \rho \\ -c \end{pmatrix}. \tag{36}$$

[2] Refer to equations (42) and (43) in [5] for a proof of this identity.

Considering the inversion transformation, it is easy to see that

$$G = (|c|^2 + \rho^2)^{-1/2}, \qquad \Lambda = G\begin{pmatrix} c & \rho \\ \rho & -c^\dagger \end{pmatrix} \tag{37}$$

is a (nonsingular-gauge) solution of equation (11). So we obtain

$$\tilde{a} = \frac{1}{|c|^2 + \rho^2}\begin{pmatrix} 1_2 \otimes \rho \\ -c^\dagger \end{pmatrix}, \tag{38}$$

which looks as if the inversion transformation is performed in a five-dimensional Euclidean space, with ρ the fifth dimension. This interesting phenomenon is also related to the well-known fact that the single-instanton solution has an $O(5)$ invariance group [6].

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